On the expectation of the First Theorem in the rough path theory

Terry Lyons (University of Oxford)
and Keisuke Hara (Ritsumeikan University)

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Abstract

We show an expectation version of the First Theorem in the rough path theory [1].

Theorem 1 *Let* $X_{s,t}^{(n)}$ *be a multiplicative functional in* $T^{(n)}$ *(for each path) whose expectation is controlled by a control function* $\omega(s,t)$ *as follows:*

$$E\left[\left\|X_{s,t}^{i}\right\|^{p/i}\right] \leq \omega(s,t)^{\theta}, \quad 1 \leq i \leq p, \tag{1}$$

for some $\theta > 1$ *and* p *such that* $[p] = n$*. Then, there exists a multiplicative extension* $X_{s,t}^{(m)}$ *to* $T^{(m)}$ *for* $m > n$ *that satisfies the following estimates,*

$$E\left[\left\|X_{s,t}^{i}\right\|^{p/i}\right] \leq C(i,p)\,\omega(s,t)^{\theta}, \quad i > p, \tag{2}$$

for constants $C(i,p)$*, and this extension is unique almost surely.*

Proof. First we show the existence by induction. Fix $m \geq [p]$. We suppose that a multiplicative functional

$$X_{s,t}^{(m)} = (1, X_{s,t}^{1}, \ldots, X_{s,t}^{[p]}, X_{s,t}^{[p]+1}, \ldots, X_{s,t}^{m})$$

satisfies $X_{u,v}^{(m)} \in T^{(m)}$, $X_{u,w}^{(m)} = X_{u,v}^{(m)} \otimes X_{v,w}^{(m)}$, and

$$E\left[\left\|X_{u,v}^{i}\right\|^{p/i}\right] \leq c_i \omega(s,t)^{\theta},$$

for all $u < v$ and $i \leq m$.

We want to construct a multiplicative functional $X_{s,t}^{(m+1)}$ satisfying the same estimates.

Consider

$$\widetilde{X}_{s,t} = (1, X_{s,t}^{1}, \ldots, X_{s,t}^{(m)}, \mathbf{0}).$$

Note that $\widetilde{X}_{s,t}$ is in $T^{(m+1)}$, but it is not multiplicative. Fix a dissection $D = \{s \leq t_1 \leq \cdots \leq t_{i-1} \leq t\}$ of $[s,t]$ and define

$$\widetilde{X}^D_{s,t} = \widetilde{X}_{s,t_1} \otimes \widetilde{X}_{t_1,t_2} \otimes \cdots \otimes \widetilde{X}_{t_{i-1},t},$$

using the multiplication in $T^{(m+1)}$. We will show the existence of the limit as the mesh of D goes to 0. If it exists, we can show the multiplicative property by taking the limit of the relation:

$$\widetilde{X}^D_{s,u} = \widetilde{X}^{D\cap[s,t]}_{s,t} \otimes \widetilde{X}^{D\cap[t,u]}_{t,u}.$$

Our task is to study the last term $(\widetilde{X}^D_{s,t})^{(m+1)}$, since the term $(\widetilde{X}^D_{s,t})^i = X^i_{s,t}$ is multiplicative for all $i \leq m$. We want to show

$$E\left[\left\|(\widetilde{X}^D_{s,t})^{(m+1)}\right\|^{p/(m+1)}\right] \leq c_{m+1}\omega(s,t)^\theta.$$

Consider another dissection D'. Then we have the triangle inequality (Note that $p/(m+1) < 1$):

$$\begin{aligned} E\left[\left\|(\widetilde{X}^D_{s,t})^{(m+1)}\right\|^{p/(m+1)}\right] \quad &\leq \quad E\left[\left\|(\widetilde{X}^D_{s,t} - \widetilde{X}^{D'}_{s,t})^{(m+1)}\right\|^{p/(m+1)}\right] \\ &\quad +E\left[\left\|(\widetilde{X}^{D'}_{s,t})^{(m+1)}\right\|^{p/(m+1)}\right]. \end{aligned}$$

By Lemma 2.2.1, we can choose j such that

$$\omega(t_{j-1}, t_{j+1}) \leq \begin{cases} \frac{2}{r-1}\omega(s,t) & r \geq 3, \\ \omega(s,t) & r = 2, \end{cases}$$

for the dissection $D = \{s = t_0 < t_1 < \cdots < t_r = t\}$. Now let $D' = D \backslash \{t_j\}$ with this point t_j chosen above carefully.

By algebraic computations with the multiplicative property, we have

$$\widetilde{X}^D_{s,t} - \widetilde{X}^{D'}_{s,t} = \left(0, \ldots, \mathbf{0}, \sum_{i=1}^{m} X^i_{t_{j-1},t_j} \otimes X^{(m+1)-i}_{t_j,t_{j+1}}\right).$$

Therefore,

$$\begin{aligned} & E\left[\left\|(\widetilde{X}^D_{s,t} - \widetilde{X}^{D'}_{s,t})^{(m+1)}\right\|^{p/(m+1)}\right] \quad \text{(Note that } \tfrac{p}{m+1} < 1.) \\ \leq \quad & \sum_{i=1}^{m} E\left[\left\|X^i_{t_{j-1},t_j}\right\|^{p/(m+1)} \left\|X^{m+1-i}_{t_j,t_{j+1}}\right\|^{p/(m+1)}\right] \\ \leq \quad & \sum_{i=1}^{m} E\left[\left\|X^i_{t_{j-1},t_j}\right\|^{p/i}\right]^{\frac{i}{(m+1)}} E\left[\left\|X^{(m+1-i)}_{t_j,t_{j+1}}\right\|^{p/(m+1-i)}\right]^{\frac{(m+1-i)}{(m+1)}} \end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{i=1}^{m} (c_i \omega(t_{j-1}, t_j))^{\frac{i}{m+1}} (c_{m+1-i}\omega(t_j, t_{j+1}))^{\frac{m+1-i}{m+1}} \\
&= \sum_{i=1}^{m} c_i^{i/(m+1)} c_{m+1-i}^{(m+1-i)/(m+1)} \left(\omega(t_{j-1}, t_j)^{i/(m+1)} \omega(t_j, t_{j+1})^{(m+1-i)/(m+1)}\right)^{\theta},
\end{aligned}
$$

by Hölder's inequality and our assumption.

Note that

$$
\omega(t_{j-1}, t_j) \leq \omega(t_{j-1}, t_{j+1}), \quad \omega(t_j, t_{j+1}) \leq \omega(t_{j-1}, t_{j+1}),
$$

and take c_{m+1} such that

$$
\sum_{i=1}^{m} c_i^{i/(m+1)} c_{m+1-i}^{(m+1-i)/(m+1)} \leq c_{m+1}.
$$

Then we have

$$
E\left[\left\|(\widetilde{X}_{s,t}^{D} - \widetilde{X}_{s,t}^{D'})^{(m+1)}\right\|^{p/(m+1)}\right] \leq c_{m+1}\omega(t_{j-1}, t_{j+1})^{\theta}.
$$

Recall that we chose t_j carefully. Successively dropping points, we have that

$$
\begin{aligned}
E\left[\left\|(\widetilde{X}_{s,t}^{D} - \widetilde{X}_{s,t}^{D'})^{(m+1)}\right\|^{p/(m+1)}\right] &\leq c_{m+1}\left\{1 + \sum_{r=3}^{\infty}\left(\frac{2}{r-1}\right)^{\theta}\right\}\omega(s,t)^{\theta} \\
&\leq C(\theta)\, c_{m+1}\, \omega(s,t)^{\theta},
\end{aligned}
$$

where $C(\theta)$ is a finite constant, which is independent on the choice of the dissection D. Then we have got our basic estimate, which assure that $\widetilde{X}^D$ is a Cauchy sequence as mesh(D) goes to 0. In fact, we can follow the argument in the original "First Theorem" as follows.

Consider two dissection D and D' such that mesh$(D) < \delta$ and mesh$(D') < \delta$. We can take a common refinement $\hat{D}$ of D and D'. Fix some interval $[t_j, t_{j+1}] \in D$. Then, $\hat{D}$ breaks the interval into $t_j \leq s_{j_1} \leq \cdots \leq s_{j_r} = t_{j+1}$, say $\hat{D}_j$. Now we know that

$$
\begin{aligned}
E\left[\left\|\widetilde{X}^{\hat{D}_j} - \widetilde{X}\right\|^{p/(m+1)}\right] &\leq c\sum_j \omega(t_j, t_{j+1})^{\theta} \\
&\leq c\left(\sum_j \omega(t_j, t_{j+1})\right) \cdot \max_j \omega(t_j, t_{j+1})^{\theta-1} \\
&\leq c\omega(s,t) \max_j \omega(t_j, t_{j+1})^{\theta-1},
\end{aligned}
$$

which is independent on $\hat{D}$. Then it converges uniformly to 0 as mesh$(D) \to 0$. Therefore, the triangle inequality

$$
E\left[\left\|\widetilde{X}^{D} - \widetilde{X}^{D'}\right\|^{p/(m+1)}\right] \leq E\left[\left\|\widetilde{X}^{D} - \widetilde{X}^{\hat{D}_j}\right\|^{p/(m+1)}\right] + E\left[\left\|\widetilde{X}^{\hat{D}_j} - \widetilde{X}^{D'}\right\|^{p/(m+1)}\right]
$$

shows that we have established the Cauchy sequence.

Only difference between the original estimate and ours is that our limit is in the sense of

$$E\left[\left\|\widetilde{X}^D - \widetilde{X}^{D'}\right\|^{p/(m+1)}\right] \to 0 \quad \text{as mesh}(D) \text{ and mesh } (D') \text{ go to } 0.$$

But, since $p/(m+1) < 1$ and so we have the triangle inequality, we can mimic the usual argument for completeness of γ-th integrable function space. In this sense of the limit, we have also

$$\lim \widetilde{X}^{D,[s,t]} = \lim(\widetilde{X}^{D,[s,u]} \otimes \widetilde{X}^{D,[u,t]}) = (\lim \widetilde{X}^{D,[s,u]}) \otimes (\lim \widetilde{X}^{D,[u,t]}),$$

which shows the limit is multiplicative.

Let us show the uniqueness of the extension. More precisely, we must show that any two multiplicative functional X_{st} and Y_{st} agree almost surely, if they agree up to m-th degree (i.e., $X^i_{st} = Y^i_{st}$, $i \leq m$) and if their expectations satisfy our condition in our theorem.

Set $\Psi_{st} = X^{m+1}_{st} - Y^{m+1}_{st}$. By Lemma 2.2.3, we know that Ψ_{st} is additive:

$$\Psi_{s,t} + \Psi_{t,u} = \Psi_{s,u}.$$

Now we have that

$$E\left[\|\Psi_{0t} - \Psi_{0s}\|^{p/(m+1)}\right] = E\left[\|\Psi_{st}\|^{p/(m+1)}\right] \leq c\,\omega(s,t)^\theta,$$

for a constant c and $\theta > 1$. Note that

$$\omega(s,t) \leq \omega(0,t) - \omega(0,s),$$

and that ω is an increasing function. So we can apply time change to have

$$E\left[\|\Psi_{0t'} - \Psi_{0s'}\|^{p/(m+1)}\right] \leq c\,(t' - s')$$

in this time scale. The key is that any time change does not change the variational norm.

Now, by the dyadic argument in Hambly-Lyons ([2]), we have

$$\begin{aligned} & \sup_D \sum_j \|\Psi_{0,t'_{j+1}} - \Psi_{0,t'_j}\|^{p/(m+1)} \\ \leq \quad & 2\sum_{n=1}^{\infty} \sum_{0 \leq k < 2^n} \|\Psi_{0,(k+1)/2^n} - \Psi_{0,k/2^n}\|^{p/(m+1)} \end{aligned}$$

with the triangle inequality ($p/(m+1) < 1$). Taking the expectation,

$$E\left[\sup_D \sum_j \|\Psi_{0,t_{j+1}} - \Psi_{0,t_j}\|^{p/(m+1)}\right]$$

$$
\begin{aligned}
&\leq \quad 2\sum_{n=1}^{\infty}\sum_{0\leq k<2^n} E\left[\|\Psi_{0,(k+1)/2^n}-\Psi_{0,k/2^n}\|^{p/(m+1)}\right] \\
&\leq \quad 2c\sum_{n=1}^{\infty}\sum_{0\leq k<2^n}\left|\frac{k+1}{2^n}-\frac{k}{2^n}\right|^{\theta} \\
&= \quad 2c\sum_{n=1}^{\infty}\sum_{0\leq k<2^n}\frac{1}{2^{n\theta}} = 2c\sum_{n=1}^{\infty}\frac{1}{2^{n(\theta-1)}} < \infty.
\end{aligned}
$$

Therefore, we have almost surely

$$
\|\Psi_{0,t}\|_{p/(m+1)-\mathrm{var}} < \infty,
$$

on $[0,1]$. We conclude that $\Psi_{0,t}\equiv 0$, so $X=Y$ almost surely. ■

References

[1] Terry J. Lyons. *Differential equations driven by rough signals*, Revista Mathematica Iberoamericana, **14**, No. 2, pp.215–310, 1998.

[2] B. M. Hambly and T. J. Lyons. *Stochastic area for Brownian motion of the Sierpinski gasket*, The Annals of Probability, **26**, No.1, pp.132–148, 1998.