On Fourier transform of rough paths

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Contents

1 A problem, an answer, and two ideas

We consider the Fourier transform type integral of a rough path. Our main interest is the rough path property of the integral.

Let $F : \mathbf{R} \to \mathbf{R}$ be a continuous path with global finite p-variation controlled by a control function ω_F for $1 \leq p$, i.e.,

$$|F(t) - F(s)|^p \leq \omega_F(s,t) < M < +\infty$$

for any $-\infty < s < t < \infty$ and a real number M.

Let us define the Fourier type integral $G^\theta(t)$ by $dG^\theta = e^{i\theta t} dF_t$, that is,

$$G^\theta(t) - G^\theta(s) = \int_s^t e^{i\theta t} dF_t \quad \text{for } s < t.$$

Then $G^\theta(t)$ is a p-rough path defined locally on each finite interval. Take $p' \geq p$ and let $\omega_{G^\theta} = \omega_{G^\theta,(p',\theta)}$ be the control function for the p'-variation of G^θ.

The problem is

Problem 1 *Is $\omega_{G^\theta,(p',\theta)}$ globally finite for almost every θ? Or more directly,*

$$\omega_{G^\theta,(p',\theta)}(-\infty, +\infty) \textit{ is finite for a.e. } \theta \textit{ ?}$$

If we imagine the pointwise convergence of Fourier transform, this problem is far from obvious. We will show the following theorem as a partial result:

Theorem 2 *Suppose that* $1 \leq p < 2$ *and* F *is a geometric rough path with* p*-variation controlled by* ω_F*, where* $\omega_F(-\infty, \infty) < M < \infty$ *for a constant* M*. If* $p < p' < 2$*, then the* p'*-variation* $\omega_{G^\theta,(p',\theta)}$ *of* G^θ *is finite for almost every* θ*.*

Our idea to show the theorem is two-fold: to estimate of the expectation of the integral with Young's idea (Section 2) and to apply Hambly-Lyons' dyadic argument with the estimate and some modification (Section 3).

First we will get an estimate for the second moment of the integral $\int_S^T e^{i\theta t} dF_t$ with Gaussian θ, that is,

$$\mathbb{E}\left[\left|\int_S^T e^{i\theta t} dF_t\right|^2\right] = \int_S^T \int_S^T e^{-(t-s)^2/2} dF_s dF_t \leq C\omega_F(S,T)^{2/p}.$$

This estimate is not innocent as it looks. Actually we will use Young's argument carefully in the real plane $\mathbb{R}^2$ according to [2] (Lyons 1981).

Secondly we use this estimate to apply Hambly-Lyons' *dyadic argument* (see [1] (Hambly-Lyons 1998)). Roughly saying, this argument is to paste the estimates on dyadic intervals like $[k/2^n, (k+1)/2^n] (k = 0, \ldots, 2^n - 1)$ in some clever way to get an estimate on the whole interval. But we need to modify this argument a little because Hambly-Lyons' trick works on the unit interval $[0, 1]$ and our integral is defined on the real line $\mathbb{R}$ on the other hand. For this, we will take a time change $\rho : [0, 1] \to \mathbb{R}$ suitably (according to ω_F) and we will check that our estimate for

$$\mathbb{E}\left[\left|\int_{\rho(k/2^n)}^{\rho((k+1)/2^n)} e^{i\theta t} dF_t\right|^2\right]$$

is sharp enough for the dyadic argument to work in the following manner:

$$\begin{aligned}
\mathbb{E}\left[\left|\int_{\rho(k/2^n)}^{\rho((k+1)/2^n)} e^{i\theta t} dF_t\right|^{p'}\right] &\leq \mathbb{E}\left[\left|\int_{\rho(k/2^n)}^{\rho((k+1)/2^n)} e^{i\theta t} dF_t\right|^2\right]^{p'/2} \\
&\leq (\text{const}) \cdot \omega_F\left(\rho\left(\frac{k+1}{2^n}\right), \rho\left(\frac{k}{2^n}\right)\right)^{(2/p)\cdot(p'/2)} \\
&\leq (\text{const}) \cdot \omega_F\left(\rho\left(\frac{k+1}{2^n}\right), \rho\left(\frac{k}{2^n}\right)\right)^{p'/p}.
\end{aligned}$$

Now using the fact that we properly chose the time change ρ according to the control ω_F and that $p < p' < 2$, we will finally get the result thanks to Hambly-Lyons' dyadic argument.

2 Controlling the second moment

2.1 the second moment when θ is Gaussian

We want to use the expectation for θ to estimate $\int_S^T e^{i\theta t} dF_t$. It is natural to choose the Gaussian distribution for θ (with the mean 0 and the variance σ^2) because this should minimize the effect of the tail of large θ and also because it allows us to get a nice representation of the second moment of $\int_S^T e^{i\theta t} dF_t$.

Suppose F is a geometric p-rough path which is constant outside the interval $[S, T]$. Then $\int_S^t e^{i\theta u} dF_u$ is well defined as a geometric rough path defined for each θ. We concern the p-variation estimate of this integral. But we approach to it with the second moment. Let us denote the expectation with respect to the Gaussian variable θ by $\mathbb{E}^\theta$. We have the following lemma.

Lemma 3

$$\mathbb{E}^\theta \left[\left| \int_S^T e^{i\theta u} dF_u \right|^2 \right] = \int_S^T \int_S^T e^{-(t-s)^2\sigma^2/2} dF_s dF_t.$$

Proof. By Fubini's theorem and simple computations, we have

$$\begin{aligned} \mathbb{E}^\theta \left[\left| \int_S^T e^{i\theta s} dF_s \right|^2 \right] &= \mathbb{E}^\theta \left[\int_S^T e^{i\theta s} dF_s \int_S^T e^{-i\theta t} dF_t \right] \\ &= \int_S^T \int_S^T \mathbb{E}^\theta \left[e^{i\theta(s-t)} \right] dF_s dF_t \\ &= \int_S^T \int_S^T e^{-(t-s)^2\sigma^2/2} dF_s dF_t, \end{aligned}$$

where the last equation is a simple Gaussian property that $\mathbb{E} e^{i\theta t} = e^{-t^2\sigma^2/2}$. ■

Remark 4 *The simple case $\sigma^2 = 1$ is enough to show our result. But this little generalization would be useful when we go further.*

2.2 the estimate in the off-diagonal box

Now we want to estimate the Gauss type integral $\int_S^T \int_S^T e^{-(t-s)^2} dF_t dF_s$. Though it might seem innocent, this procedure is not easy. Since the diagonal seems to be needed special treatment, we first show the following estimate in the off-diagonal box.

Lemma 5 *Suppose that F and G are p-rough paths controlled by ω_F and ω_G respectably, which are both globally finite, i.e., $\omega_F(-\infty, \infty), \omega_G(-\infty, \infty) < M < \infty$ for some constant M. Suppose futher that $\omega_F(0, \infty) = \omega_G(-\infty, 0) = 0$ (,* *which means that F is constant on the right half line and G is constant on the left half line). Then, we have the following estimate:*

$$\int_{\mathbb{R}}\int_{\mathbb{R}} e^{-(t-s)^2\sigma^2/2} dF_s dG_t \leq C\cdot\omega_F(-\infty,\infty)^{1/p}\omega_G(-\infty,\infty)^{1/p},$$

for a constant $C>0$.

Since the integral above does not contain the diagonal $\{s=t\}$, the result is a direct consequence of the following theorem by T.J.Lyons [2]:

Theorem 6 (T.J.Lyons (1981)) *Let $1/p+1/q>1$. If f is of finite q-variation controlled by ω_f and g is of finite p-variation controlled by ω_g, then the integral $z_t=\int_0^t f(s)dg(s)$ has its p-variation controlled by*

$$(\|f\|_\infty + C_{1/p+1/q}\,\omega_f(0,t)^{1/q})^p\omega_g(0,t).$$

Therefore, in particular if f has bounded 1-variation and we take $q=1$, the variation is at most $(\|f\|_\infty + C_{1/p+1}\,\omega_f(0,t))^p\omega_g(0,t)$. Using this fact twice, we can give a proof of the lemma above as follows.

Proof. (Proof of Lemma 5).

Since F_s has bounded variation, it has a limit as $s\to-\infty$. Therefore, we can apply a suitable time change $s\mapsto -s$ to make it run backwards. Let us denote the backward function by $\tilde{F}_s$. Note that this time change does not change the shape of the path, so it is a p-rough path again. Then, we have

$$\begin{aligned}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{-(t-s)^2\sigma^2/2} dF_s dG_t &= \int_{t>0}\int_{s<0} e^{-(t-s)^2\sigma^2/2} dF_s dG_t\\ &= \int_{t>0}\int_{s>0} e^{-(t+s)^2\sigma^2/2} d\tilde{F}_s dG_t\\ &\leq C\cdot\omega_{\tilde{F}}(-\infty,\infty)^{1/p}\omega_G(-\infty,\infty)^{1/p}\\ &= C\cdot\omega_F(-\infty,\infty)^{1/p}\omega_G(-\infty,\infty)^{1/p},\end{aligned}$$

if we can show the inequality above in the middle.

Therefore, our task is to estimate

$$\int_0^\infty\left(\int_0^\infty e^{-(t+s)^2\sigma^2/2} dX_s\right) dY_t$$

for a p-rough paths X,Y controlled by global bounded control functions ω_X,ω_Y respectively defined on $[0,\infty]$.

According to Theorem 6, It is enough to control the uniform norm and 1-variation of

$$Z(t)=\int_0^\infty e^{-(t+s)^2\sigma^2/2} dX_s.$$

First we check the uniform bound. Certainly the integral above makes sense since the integrand $s\mapsto\exp(-(t+s)^2\sigma^2/2)$ is bounded and has globally bounded

1-variation thanks to Theorem 6 above. More precisely, we have the Young integral bound at $t>0$ and taking the supremum over $t>0$, we get

$$\begin{aligned}|Z(t)| = \left|\int_0^\infty e^{-(t+s)^2\sigma^2/2} dX_s\right| &\leq \omega_X(0,\infty)^{1/p}(e^{-t^2\sigma^2/2} + C_{1/p+1}e^{-t^2\sigma^2/2}) \\ &\leq \omega_X(0,\infty)^{1/p}(1+C_{1/p+1})\, e^{-t^2\sigma^2/2} \\ &\leq \omega_X(0,\infty)^{1/p}(1+C_{1/p+1}),\end{aligned}$$

because the supremum norm and the 1-variational norm of $s \mapsto e^{-(s+t)^2\sigma^2/2}$ are less than or equal to $e^{-(t+0)^2\sigma^2/2}$. So we have got the uniform bound of $|Z(t)|$.

Next, to estimates the 1-variational norm for $Z(t)$, we study the derivative of $Z(t)$:

$$Z(t)' = \int_0^\infty -(t+s)\sigma^2\, e^{-(t+s)^2\sigma^2/2} dX_s.$$

Let us denote the integrand by $-z(t)$, i.e.,

$$z(t) = (t+s)\sigma^2\, e^{-(t+s)^2\sigma^2/2} \quad (t>0),$$

to see the behaviour closely. Simple calculation shows

$$z'(t) = \sigma^2 e^{-(t+s)^2\sigma^2/2}\left(1-\sigma^2(t+s)^2\right).$$

Therefore, we have two cases. If $0 \leq s \leq 1/\sigma$, the maximum of $z(t)$ occurs exactly once when $(t+s)^2\sigma^2 = 1$ and the maximal value is $z(1/\sigma - s) = 1/\sigma \cdot \sigma^2 e^{-1/2} = \sigma e^{-1/2}$. On the other hand, if $s > 1/\sigma$, the function $z(t)$ is monotone and the maximal value is $z(0) = s\sigma^2 e^{-s^2\sigma^2/2}$. This observation with Theorem 6 allows us to estimate of the integral $Z'(t)$ as follows.

$$\begin{aligned}|Z'(t)| &= \left|\int_0^\infty -(t+s)\sigma^2 e^{-(t+s)^2\sigma^2/2} dX_s\right| \\ &\leq \begin{cases} \omega_X(0,\infty)^{1/p}(1+C_{1/p+1})\sigma e^{-1/2}, & \text{if } t \leq 1/\sigma, \\ \omega_X(0,\infty)^{1/p}(1+C_{1/p+1}) t\sigma^2 e^{-t^2\sigma^2/2}, & \text{if } t > 1/\sigma. \end{cases}\end{aligned}$$

Since $|Z(v)-Z(u)| = \left|\int_u^v Z'(t)dt\right| \leq \int_u^v |Z'(t)|dt$, we can get the bound of the 1-variation of $Z(t)$ with the estimate above for $|Z'(t)|$ as follows. For any sequence $0 \leq t_0 < t_1 < t_2 < \cdots < \infty$, we have

$$\begin{aligned}\sum_j |Z(t_{j+1}) - Z(t_j)| &\leq \sum_{j<k} |Z(t_{j+1}) - Z(t_j)| + |Z(1/\sigma) - Z(t_k)| \\ &\quad + |Z(t_{k+1}) - Z(1/\sigma)| + \sum_{j\geq k+1} |Z(t_{j+1}) - Z(t_j)| \\ &\leq \omega_X(0,\infty)^{1/p}(1+C_{1/p+1})\sigma e^{-1/2}\frac{1}{\sigma} \\ &\quad + \omega_X(0,\infty)^{1/p}(1+C_{1/p+1})\sigma^2 \int_{1/\sigma}^\infty te^{-t^2\sigma^2/2}dt \\ &= 2\omega_X(0,\infty)^{1/p}(1+C_{1/p+1})e^{-1/2}.\end{aligned}$$

Therefore, the 1-variation of $Z(t) = \int_0^\infty \exp(-(t+s)^2\sigma^2/2)dX_s$ is at most $2\omega_X(0,\infty)^{1/p}(1+C_{1/p+1})e^{-1/2}$.

Therefore, we have the estimates for the supremum norm and 1-variation norm of $Z(t)$. Finally applying Theorem 6 to $\int_0^\infty Z(t)dY_t$ again, we get

$$\begin{aligned}
\left|\int_0^\infty \int_0^\infty e^{-(t+s)^2\sigma^2/2} dX_s dY_t\right| &= \left|\int_0^\infty Z(t)dY_t\right| \\
&\leq \omega_Y^{1/p}(0,\infty)\left((1+C_{1/p+1})\omega_X^{1/p}(0,\infty) + C_{1/p+1}(1+C_{1/p+1})2e^{-1/2}\omega_X^{1/p}(0,\infty)\right) \\
&\leq C\omega_X^{1/p}(0,\infty)\omega_Y^{1/p}(0,\infty)
\end{aligned}$$

for some constant C. We have finished the proof. ∎

2.3 The essential trick and the estimate in $\mathbb{R}^2$

Now we apply our estimate on the off-diagonal box to the whole integral by using Young's trick as developed in [2] (Lyons 1981). In usual Young's argument to define Young's integral, we choose carefully a removing point s_j in succession from the partition $s_0 < s_1 < \cdots < s_n$ of the interval on which the integral is defined. Roughly saying, here we use the similar idea in the 2 dimension.

We prepare the following definition for the procedure.

Definition 7 *Let $D = \{s_0 = -\infty < s_1 < s_2 < \cdots < s_r = +\infty\}$ be a finite partition of $\mathbb{R}$ and consider the domain $\Delta(S,T) = \{(s,t) : S \leq s < t \leq T\}$, and the domain $\Delta^D = \Delta(-\infty,\infty) \setminus \cup_{i=1,\dots,r}\Delta(s_{i-1},s_i)$. We define the D-approximate integral*

$$I^D = \int\int_{\Delta^D} e^{-(t-s)^2\sigma^2/2} dF_s dG_t.$$

We show the next theorem with Young's trick and the key lemma (Lemma 5) deduced in the last subsection.

Theorem 8 *Suppose that $1 \leq p < 2$. Then the following estimate holds.*

$$|I^D| \leq C\omega_F(-\infty,\infty)^{1/p}\omega_G(-\infty,\infty)^{1/p},$$

for some constant C.

Proof. Without loss of generality, we can rescale F and G such that $\omega_F(-\infty,\infty) = \omega_G(-\infty,\infty)$. Let $\omega = \omega_F + \omega_G$. (So ω controls the both of F and G.) If $r = 1$, i.e., D is $(-\infty,\infty)$, then Δ^D is empty , $|I^D| = 0$, and no problem. Otherwise, there is an i such that

$$\omega(s_{i-1},s_{i+1}) \leq \begin{cases} 2\frac{\omega(-\infty,\infty)}{r-2} & \text{if } r > 2, \\ \omega(-\infty,\infty) & \text{if } r = 2. \end{cases}$$

Let $D' = D \setminus \{s_i\}$. Then

$$\begin{aligned} I^D - I^{D'} &= \int\int_{\Delta^D \setminus \Delta^{D'}} e^{-(t-s)^2\sigma^2/2} dF_s dG_t \\ &= 2\int\int_{[s_{i-1},s_i]\times[s_i,s_{i+1}]} e^{-(t-s)^2\sigma^2/2} dF_s dG_t. \end{aligned}$$

(See Figure 1.)

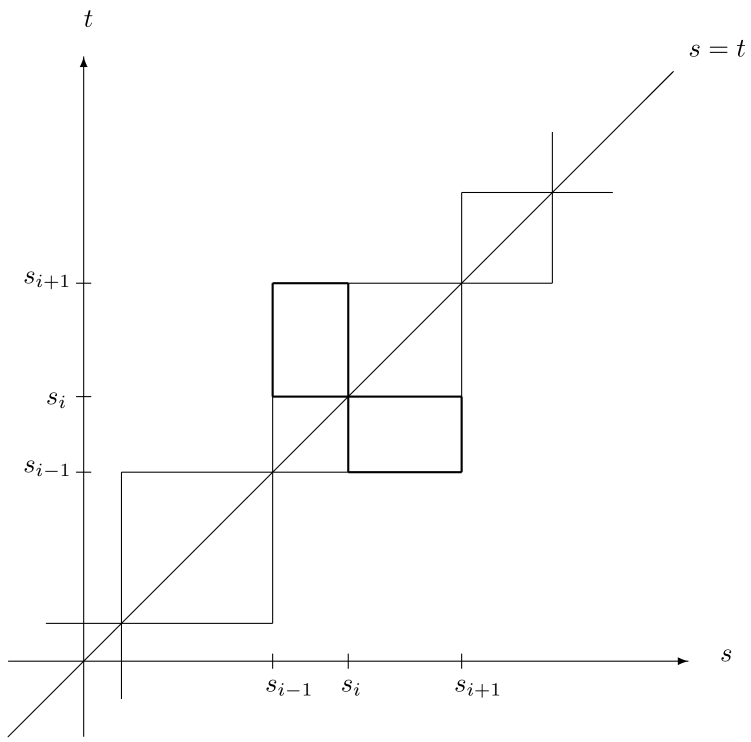

Figure 1: Young's trick in 2 dimension

Now we apply Lemma 5 to this integral on the off-diagonal box and deduce that

$$\begin{aligned} |I^D - I^{D'}| &\leq \left| \int 2 \int_{[s_{i-1},s_i]\times[s_i,s_{i+1}]} e^{-(t-s)^2\sigma^2/2} dF_s dG_t \right| \\ &\leq C\omega_F(s_{i-1}, s_i)^{1/p} \omega_G(s_i, s_{i+1})^{1/p} \\ &\leq C' \left(\frac{\omega(-\infty, \infty)}{r-2} \right)^{2/p}. \end{aligned}$$

Since $p < 2$, we can sum up these estimates to get

$$|I^D| \leq C' 2^{2/p} \left(1 + \sum_{r=2}^{\infty} \left(\frac{2}{r}\right)^{2/p}\right) \omega_F(-\infty,\infty)^{1/p} \omega_G(-\infty,\infty)^{1/p},$$

uniformly in D.

■

The following is the direct consequence of this uniform bound.

Corollary 9 *If F and G are of uniformly bounded p-variation for $1 \leq p < 2$ on $\mathbb{R}$, the integral $\int_{\mathbb{R}} \int_{\mathbb{R}} \exp(-(t-s)^2\sigma^2/2) dF_s dG_t$ makes sense and it has the bound*

$$\left| \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-(t-s)^2\sigma^2/2} dF_s dG_t \right| \leq (const.) \omega_F(-\infty,\infty)^{1/p} \omega_G(-\infty,\infty)^{1/p}.$$

Proof. Since we have already the uniform bound, the rest is a standard approximation argument to get the Cauchy sequence. Let F^R and G^R be the truncated path of F, G on $[-R, R]$. More precisely, $F^R = F$ on $[-R, R]$ and $F^R = 0$ on $[-R, R]^c$. Notice that F^R converges to F in p-variation, because the variation of F^R is monotone and bounded as $R \to \infty$. The integral $\int\int \exp(-(t-s)^2\sigma^2/2) dF^R dG^R$ is well-defined as a usual Young's integral (here we use that $p < 2$). On the other hand, the theorem above assures

$$\begin{aligned} & \sup_D \left| \int\int_{\Delta^D} e^{-(t-s)^2\sigma^2/2} dF^R_s dG^R_t - \int\int_{\Delta^D} e^{-(t-s)^2\sigma^2/2} dF_s dG_t \right| \\ \leq \quad & (\text{const.}) \|F^R - F\|_p \|G^R - G\|_p. \end{aligned}$$

Therefore, the Cauchy sequence for two partitions D_1 and D_2

$$\left| \int\int_{\Delta^{D_1}} e^{-(t-s)^2\sigma^2/2} dF_s dG_t - \int\int_{\Delta^{D_2}} e^{-(t-s)^2\sigma^2/2} dF_s dG_t \right|$$

converges as the mesh of the partitions goes to zero considering the triangle inequality with the approximation above. ■

3 Applying Hambly-Lyons' dyadic argument

3.1 Recall Hambly-Lyons' dyadic argument

Let us recall the Hambly-Lyons's dyadic argument. The following theorem proved by Hambly and Lyons in ([1] 1998) shows the power in condensed form. We show only the statement.

Lemma 10 (B.Hambly and T.J.Lyons (1998)) *Suppose that $(X^k_{s,t})$ is a continuous multiplicative functional on $\Delta(0,1)$. Then there exists a constant $C(p)$ such that $(X^k_{s,t})$ will have finite p-variation on $[0,1]$ if*

$$\sum_{n=0}^{\infty} n^{C(p)} \sum_{k=1}^{2^n} \max_{l \leq p} \left| X^l\left(\frac{k}{2^n}\right) - X^l\left(\frac{k+1}{2^n}\right) \right|^{p/l} < \infty.$$

We will use this lemma to deduce the finiteness of p'-variation of G^θ with the estimate of the expectation. The hypothesis of the lemma is clearly satisified for almost all choice of a random X if

$$\mathbb{E}\left[\sum_{n=0}^{\infty} n^{C(p)} \sum_{k=1}^{2^n} \max_{l\leq p} \left| X^l\left(\frac{k}{2^n}\right) - X^l\left(\frac{k+1}{2^n}\right)\right|^{p/l}\right] < \infty,$$

which is easily verified almost surely if the random variable above has finite expectation.

But our object G^θ is defined on $\mathbb{R}$ that is a non compact interval instead of the compact interval $[0,1]$. Therefore we need a little work to adjust the situation.

3.2 A proper time change

To adjust our functions to Hambly-Lyons' lemma, we prepare a suitable time change. Our time change should be not only a map from $[0,1]$ to $\mathbb{R}$, but also go well with a control function.

Lemma 11 *If ω is a control function where $\omega(-\infty,\infty) < M < \infty$ for a constant M, then there exists a continuous strictly increasing function $\rho : [0,1] \to [-\infty,\infty]$ with the property that*

$$\omega(\rho(u), \rho(v)) \leq M|u-v|.$$

Proof. Let $\tau(t) = \omega(-\infty, t)$, then τ is continuous and increasing with values in $[0, M]$. By the super-additivity of ω, we have

$$\omega(-\infty, s) + \omega(s,t) \leq \omega(-\infty, t).$$

Therefore,

$$\begin{aligned} \omega(s,t) &\leq \omega(-\infty,t) - \omega(-\infty,s) \\ &= |\omega(-\infty,t) - \omega(-\infty,s)| \\ &= |\tau(t) - \tau(s)|. \end{aligned}$$

For any $\epsilon > 0$, take any function $\tilde{\tau} : [-\infty,\infty] \to [0,\epsilon]$ that is strictly increasing, continuous, and onto. Note that

$$\omega(s,t) \leq |(\tau(t) + \tilde{\tau}(t)) - (\tau(s) + \tilde{\tau}(s))|.$$

As $\tau + \tilde{\tau}$ is one to one, continuous, and takes $-\infty$ to 0 and ∞ to $M+\epsilon$, it is onto and has a continuous inverse function. We rescale this inverse and set $\rho : [0,1] \to [-\infty,\infty]$ to be the inverse of $(\tau+\tilde{\tau})/(M+\epsilon)$. That is,

$$(\tau + \tilde{\tau})(\rho(u)) = (M+\epsilon)u.$$

In other words, we have

$$\omega(\rho(u), \rho(v)) \leq (M+\epsilon)|u-v|$$

as claimed in the lemma. ■

By this simple lemma, we can adjust our situation to Hambly-Lyons' argument. If F has finite p-variation with the control ω_F, then it has finite p'-variation for any $p' > p$ and we have

$$|F(t) - F(s)|^{p'} \leq \omega_F(s,t)^{p'/p}.$$

Since we chose $p'/p > 1$, $\omega(s,t)^{p'/p}$ is also a control function.

Under the time change ρ we prepared in lemma above, we have

$$\begin{aligned} |F(\rho(v)) - F(\rho(u))| &\leq (M+\epsilon)^{1/p}|u-v|^{1/p} \\ &\leq (M+\epsilon)^{1/p}|u-v|^{1/p-1/p'}|u-v|^{1/p'} \\ &\leq (M+\epsilon)^{1/p}|u-v|^{1/p'}, \end{aligned}$$

which is the property we need.

3.3 Putting all together

Now we have done all preparation. Finally, we apply Hambly-Lyons' lemma with a proper time change to our fundamental estimate.

Recall the consequence in Section 2 as the following truncated version. This is a trivial corollary if we consider F whose is constant outside the interval $[S,T]$.

Theorem 12 *Suppose that F is a continuous path on $\mathbb{R}$ with bounded p-variation for $1 \leq p < 2$. Let ω_F be the control function that $\omega(-\infty,\infty) < M < \infty$ for some constant M. Then, for any $-\infty \leq S < T \leq \infty$, we have*

$$\mathbb{E}^\theta\left(\left|\int_S^T e^{i\theta u} dF_u\right|^2\right) = \left|\int_S^T \int_S^T e^{-(t-s)^2\sigma^2/2} dF_s dF_t\right| \leq D(p)\omega_F(S,T)^{2/p},$$

where $D(p)$ is a constant depending only on p.

Therefore, we have the key estimate:

$$\begin{aligned} &\mathbb{E}^\theta\left(\left|\int_{\rho(k/2^n)}^{\rho((k+1)/2^n)} e^{i\theta t} dF_u\right|^2\right)^{p'/2} \\ &= \left|\int_{\rho(k/2^n)}^{\rho((k+1)/2^n)} \int_{\rho(k/2^n)}^{\rho((k+1)/2^n)} e^{-(t-s)^2\sigma^2/2} dF_s dF_t\right|^{p'/2} \\ &\leq D(p)\left(\omega_F\left(\frac{k}{2^n}, \frac{k+1}{2^n}\right)\right)^{(2/p)(p'/2)} \end{aligned}$$

$$
\begin{aligned}
&\le \quad D(p)\left(\frac{\omega_F(-\infty,\infty)}{2^n}\right)^{(2/p)(p'/2)} \\
&= \quad D(p)\omega_F(-\infty,\infty)^{p'/p}\cdot 2^{-n(p'/p)},
\end{aligned}
$$

where we used the super-additivity of ω_F in the second inequality. Here it is very crucial to get the order $2^{-n(p'/p)}$ for $p' > p$. This order assures us to satisfy the assumption of Hambly-Lyons' lemma in Section 3 as follows:

$$
\begin{aligned}
\mathbb{E}^\theta &\left[\sum_{n=0}^{\infty} n^{C(p')}\sum_{k=1}^{2^n}\left|G^\theta\left(\rho\left(\frac{k}{2^n}\right)\right)-G^\theta\left(\rho\left(\frac{k+1}{2^n}\right)\right)\right|^{p'}\right] \\
&\le D(p)\omega_F(-\infty,\infty)^{p'/p}\sum_{n=0}^{\infty} n^{C(p')}2^{n(1-(p'/p))} < \infty
\end{aligned}
$$

by Beppo-Levy. We have checked the assumption of Hambly-Lyons' lemma for the almost surely paths. Therefore, the time-changed path $G^\theta(\rho(\cdot))$ has uniformly finite p'-variation by Hambly-Lyons' dyadic argument. Since the variation does not depend on any time-change, we reach the conclusion.

References

[1] Hambly, B.M. and Lyons, T.J. Stochastic area for Brownian motion on the Sierpinski gasket. Ann. Prob. **26**, pp.132–148, (1998).

[2] Lyons, T.J. research letters paper (1981).